ELASTIC EQUILIBRIUM OF AN INFINITE BODY WEAKENED BY AN EXTERNAL CIRCULAR CRACK

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1. Introduction. This study presents an exact solution of the first fundamental problem of the theory of elasticity for infinite space, containing a flat cut covering the outside of a circle (Fig. 1).

When solving problems of this type, it is convenient to introduce side by side with rectangular coordinates (x, y, z) toroidal coordinates $(0 \leqslant \alpha < \infty, -\pi \leqslant \beta \leqslant +\pi, 0 \leqslant \phi < 2\pi)$, using relations

$$x = \frac{a \operatorname{sh} \alpha \cos \varphi}{\operatorname{ch} \alpha + \cos \beta}, \qquad y = \frac{a \operatorname{sh} \alpha \sin \varphi}{\operatorname{ch} \alpha + \cos \beta}, \qquad z = \frac{a \sin \beta}{\operatorname{ch} \alpha + \cos \beta} \tag{1.1}$$

In this case the edges of the cut are coordinate surfaces $\beta = \pm \pi$, the inside of the circle of radius a is the surface $\beta = 0$, and the dividing circumference $r = a$ is the line $\alpha = \infty$.

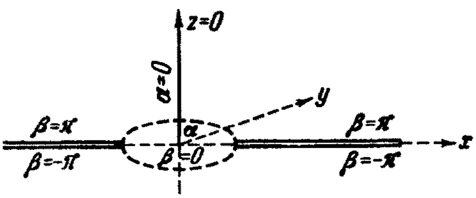

Fig. 1.

To solve this problem we will make use of the Papkovich-Neuber representation of elastic displacements (u, v, w) through four harmonic functions $\Phi_k (k = 0, 1, 2, 3)$

$$2\mu u = -\frac{\partial F}{\partial x} + 4(1-\nu)\Phi_1, \qquad 2\mu v = -\frac{\partial F}{\partial y} + 4(1-\nu)\Phi_2$$
$$2\mu w = -\frac{\partial F}{\partial z} + 4(1-\nu)\Phi_3, \qquad F = \Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3 \tag{1.2}$$

(μ is the shear modulus, ν is Poisson's ratio).

We will also give the expressions for the stresses (σ_z, τ_{zx}, τ_{yz}), which are known on the boundaries of the cut:

$$\sigma_z = \frac{\partial}{\partial z}[2(1-\nu)\Phi_3 - \Phi_4] + 2\nu\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y}\right) - \left(x\frac{\partial^2\Phi_1}{\partial z^2} + y\frac{\partial^2\Phi_2}{\partial z^2} + z\frac{\partial^2\Phi_3}{\partial z^2}\right)$$

$$\tau_{zx} = \frac{\partial\Phi}{\partial x} + 2(1-\nu)\frac{\partial\Phi_1}{\partial z}, \qquad \tau_{yz} = \frac{\partial\Phi}{\partial y} + 2(1-\nu)\frac{\partial\Phi_2}{\partial z} \tag{1.3}$$

$$\Phi = (1-2\nu)\Phi_3 - \Phi_4 - \left(x\frac{\partial\Phi_1}{\partial z} + y\frac{\partial\Phi_2}{\partial z} + z\frac{\partial\Phi_3}{\partial z}\right), \qquad \Phi_4 = \frac{\partial\Phi_0}{\partial z}$$

In what follows the problem will be split into two parts: one symmetric and one anti-symmetric (with respect to the surface $z = 0$); in each case it is obviously possible to lay down certain conditions at the surface $\beta = 0$ and consider the problem for the upper half-space only.

2. Symmetric problem*. In the case of a state of stress symmetric with respect to the coordinate z, the equilibrium of the upper half-space ($0 \leqslant \beta \leqslant \pi$) can be considered under the following boundary conditions:

$$w|_{\beta=0} = 0, \qquad \tau_{zx}|_{\beta=0} = 0, \qquad \tau_{yz}|_{\beta=0} = 0 \tag{2.1}$$

$$\sigma_z|_{\beta=\pi} = \sigma(\alpha, \varphi), \qquad \tau_{zx}|_{\beta=\pi} = \tau_x(\alpha, \varphi), \qquad \tau_{yz}|_{\beta=\pi} = \tau_y(\alpha, \varphi) \tag{2.2}$$

Making use of the arbitrary character of one of the harmonic functions contained in the Papkovich-Neuber solutions, to conditions (2.1)-(2.2) we will add two more additional conditions:

$$\Phi|_{\beta=0} = 0, \qquad \Phi|_{\beta=\pi} = 0 \tag{2.3}$$

Hence, such of boundary conditions (2.1)-(2.2) as are connected with shear stresses immediately enable us to formulate separate boundary conditions for harmonic functions Φ_1 and Φ_2:

$$\left.\frac{\partial\Phi_1}{\partial z}\right|_{\beta=0} = \left.\frac{\partial\Phi_2}{\partial z}\right|_{\beta=0} = 0, \quad \left.\frac{\partial\Phi_1}{\partial z}\right|_{\beta=0} = \frac{\tau_x}{2(1-\nu)}, \quad \left.\frac{\partial\Phi_2}{\partial z}\right|_{\beta=0} = \frac{\tau_y}{2(1-\nu)}. \tag{2.4}$$

Functions Φ_1 and Φ_2 can thus be considered known as a result of the solution of the Neumann problem for a half-space.

We then turn to conditions (2.3) and from these obtain the function

* The corresponding problem for the case of an internal circular crack covering the region $\beta = 0$ was solved by M. Ia. Leonov [1,2].

$$\psi = (1 - 2\nu)\,\Phi_3 - \Phi_4 \tag{2.5}$$

solving the Dirichlet problem for the half-space

$$\Delta\psi = 0, \qquad \psi|_{\beta=0} = 0, \qquad \psi|_{\beta=\pi} = \frac{1}{2(1-\nu)}(x\tau_{zx} + y\tau_{yz})_{\beta=\pi} \tag{2.6}$$

Finally, the remaining conditions $w|_{\beta=0} = 0$, $\sigma_z|_{\beta=\pi} = \sigma(\alpha, \phi)$, after the substitution of values of $\Phi_4 = (1 - 2\nu)\Phi - \psi$, lead to the mixed boundary conditions for harmonic function Φ_3:

$$\Phi_3|_{\beta=0} = 0 \tag{2.7}$$

$$\left.\frac{\partial\Phi_3}{\partial z}\right|_{\beta=\pi} = \sigma(\alpha, \varphi) + \left[\frac{\partial^2}{\partial z^2}(x\Phi_1 + y\Phi_2) - 2\nu\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y}\right) - \frac{\partial\psi}{\partial z}\right]_{\beta=\pi}$$

A method for the solution of problems with mixed boundary conditions (2.7) is given in the following Section.

3. Example. As an example we will investigate the case when external loading is represented by two normal concentrated loads P acting in opposite directions, applied at points $\alpha = \alpha_0$, $\beta = \pm\pi$, $\phi = 0$, (Fig. 2).

As the external shear stresses are equal to zero, then according to (2.4) and (2.6) $\Phi_1 = \Phi_2 = \psi = 0$, $\Phi_4 \equiv (1 - 2\nu)\Phi_3$ and the problem is reduced to solving for one harmonic function in half-space $\Phi \equiv \Phi_3$ under following boundary conditions

$$\Phi|_{\beta=0} = 0, \qquad \left.\frac{\partial\Phi}{\partial z}\right|_{\beta=\pi} = \sigma(\alpha, \varphi) \tag{3.1}$$

An effective solution of similar mixed problems can be obtained with the aid of the Mehler-Fock integral transformation [3].

Indeed, if we assume

$$\Phi = \sqrt{\operatorname{ch}\alpha + \cos\beta}\,\sum_{m=0}^{\infty} \cos m\varphi \int_0^{\infty} A_m(\tau)\,\frac{\operatorname{sh}\beta\tau}{\tau\operatorname{ch}\pi\tau}\,P^{m}_{-1/2+i\tau}(\operatorname{ch}\alpha)\,d\tau \tag{3.2}$$

where $P_\nu^m(x)$ are the associated Legendre functions, then the first condition (3.1) will be satisfied, and the second condition will lead to the equation

$$\sigma(\alpha, \varphi) = -\frac{1}{a}(\operatorname{ch}\alpha - 1)^{3/2}\sum_{m=0}^{\infty} \cos m\varphi \int_0^{\infty} A_m(\tau)\,P^{m}_{-1/2+i\tau}(\operatorname{ch}\alpha)\,d\tau \tag{3.3}$$

According to the Mehler-Fock conversion formula, we find

$$A_m(\tau) = \frac{2a}{\pi}(-1)^{m+1}\tau \operatorname{th} \pi\tau \int_0^\pi \cos m\varphi \, d\varphi \times$$

$$\times \int_0^\infty \frac{\sigma(\alpha, \varphi)}{(\operatorname{ch}\alpha - 1)^{3/2}} P_{-1/2+i\tau}^{-m}(\operatorname{ch}\alpha) \operatorname{sh}\alpha d\alpha \; (m \geq 1) \tag{3.4}$$

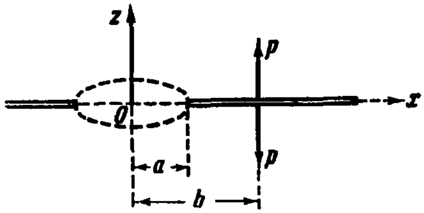

Fig. 2.

With $m = 0$, the last expression contains a factor 1/2.

For the case in question of a concentrated load we easily obtain

$$A_m(\tau) = (-1)^m \frac{P}{\pi b} \frac{\operatorname{sh}\alpha_0}{\sqrt{\operatorname{ch}\alpha_0 - 1}} \tau \operatorname{th} \pi\tau P_{-1/2+i\tau}^{-m}(\operatorname{ch}\alpha_0) \tag{3.5}$$

Thus function Φ is given by the following expression

$$\Phi = \frac{P}{\pi a} \sqrt{(\operatorname{ch}\alpha_0 - 1)(\operatorname{ch}\alpha + \cos\beta)} \sum_{m=0}^{\infty}{}' (-1)^m \cos m\varphi \times$$

$$\times \int_0^\infty \frac{\operatorname{th}\pi\tau}{\operatorname{ch}\pi\tau} \operatorname{sh}\beta\tau P_{-1/2+i\tau}^{-m}(\operatorname{ch}\alpha_0) P_{-1/2+i\tau}^{m}(\operatorname{ch}\alpha) d\tau \tag{3.6}$$

(the sign $'$ at the summation symbol indicates that the zero member of the series has the factor 1/2).

Using integral representation [4]

$$(-1)^m P_{-1/2+i\tau}^{m}(\operatorname{ch}\alpha) P_{-1/2+i\tau}^{-m}(\operatorname{ch}\alpha_0) \sqrt{\operatorname{sh}\alpha \operatorname{sh}\alpha_0} =$$

$$= \frac{2\operatorname{ch}\pi\tau}{\pi^2} \int_0^\infty Q_{m-1/2}\left(\frac{\operatorname{ch}s + \operatorname{ch}\alpha \operatorname{ch}\alpha_0}{\operatorname{sh}\alpha \operatorname{sh}\alpha_0}\right) \cos\tau s \, ds \tag{3.7}$$

as well as expansion [5]

$$\frac{1}{\sqrt{2(\operatorname{ch}u - \cos\varphi)}} = \frac{2}{\pi} \sum_{m=0}^{\infty}{}' Q_{m-1/2}(\operatorname{ch}u) \cos m\varphi \tag{3.8}$$

we can get the solution in a closed form

$$\Phi = \frac{P}{\pi^2 \rho} \operatorname{arc\,tg}\left[\frac{a}{\rho}\sqrt{\frac{b^2}{a^2} - 1}\sqrt{\frac{1 - \cos\beta}{\operatorname{ch}\alpha + \cos\beta}}\right]$$

$$\rho = \sqrt{(x - b)^2 + y^2 + z^2} \tag{3.9}$$

Let us also give the expression for stresses in the mid-section $z = 0$, $r < a$:

$$\sigma_z|_{\beta=0} = \frac{P}{\pi^2 a^2}\sqrt{\frac{b^2}{a^2}-1}\,\frac{(\operatorname{ch}\alpha_0 - 1)\operatorname{ch}^3 {}^1/_2\,\alpha}{\operatorname{ch}\alpha\operatorname{ch}\alpha_0 + 1 - \operatorname{sh}\alpha\operatorname{sh}\alpha_0\cos\varphi} \tag{3.10}$$

4. Antisymmetric problem*. In this case we have to investigate the elastic equilibrium of the upper half-space under boundary conditions

$$u|_{\beta=0} = v|_{\beta=0} = 0, \qquad \tau_{zx}|_{\beta=\pi} = \tau_x(\alpha, \varphi), \qquad \tau_{yz}|_{\beta=\pi} = \tau_y(\alpha, \varphi) \tag{4.1}$$

$$\sigma_z|_{\beta=0} = 0, \qquad \sigma_z|_{\beta=\pi} = \sigma(\alpha, \varphi) \tag{4.2}$$

Through a special choice of two additional boundary conditions, this problem also can be reduced to separated boundary problems (Dirichlet, Neumann or mixed) for four harmonic functions in half-space.

However, there are some additional difficulties. When systematically using the methods of Sections 2-3, one of the boundary conditions is satisfied to the order of one plane harmonic term.Consequently, a plane harmonic function containing unknown coefficients to be defined subsequently so as to satisfy all the conditions of the problem has to be introduced into the solution from the beginning.

Also, when carrying out the computation, some of the functions appearing on the right-hand side of the boundary conditions do not converge to zero as $a \to \infty$, and therefore do not decompose into a Mehler-Fock integral**. This difficulty can be overcome by introducing "special" solutions of Laplace's equation

$$f(\alpha, \beta, \varphi) = \sqrt{\operatorname{ch}\alpha + \cos\beta}\, e^{\pm i^1/_2\beta} \sum_{m=0}^{\infty} f_m \operatorname{th}^m \tfrac{1}{2}\alpha e^{im\varphi} \tag{4.3}$$

discontinuous on the line of separation of the boundary conditions ($a = \infty$).

In conformity with the above, we choose two additional conditions of the following form:

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* A similar problem for an internal crack has been solved by V.I. Mossakovskii [6], however he has considered a particular type of an external load, that would be expanded into a series in angle ϕ and containing a finite number of terms.

** A similar condition arises also in more complicated mixed problems of the theory of elasticity, where one also has to make use of the type of solution (4.3) (see paper [3], where case $m = 1$, relevant to the contact problem is investigated).

$$F|_{\beta=0} = \mathrm{Re} \sum_{m=0}^{\infty} F_m r^m e^{im\varphi}, \qquad \Phi|_{\beta=\pi} = 0 \tag{4.4}$$

where F_m are as yet unknown coefficients.

Then, from (4.1), there immediately follow the separate boundary conditions for the harmonic functions Φ_1 and Φ_2:

$$\Phi_1|_{\beta=0} = \frac{1}{4(1-\nu)} \frac{\partial F}{\partial x}\bigg|_{\beta=0}, \qquad \frac{\partial \Phi_1}{\partial z}\bigg|_{\beta=\pi} = \frac{\tau_x}{2(1-\nu)} \tag{4.5}$$

$$\Phi_2|_{\beta=0} = \frac{1}{4(1-\nu)} \frac{\partial F}{\partial y}\bigg|_{\beta=0}, \qquad \frac{\partial \Phi_2}{\partial z}\bigg|_{\beta=\pi} = \frac{\tau_y}{2(1-\nu)} \tag{4.6}$$

Considering Φ_1 and Φ_2 as known, from (4.2) for the harmonic function we get

$$\omega = 2(1-\nu)\Phi_3 - \Phi_4 \tag{4.7}$$

and boundary conditions

$$\frac{\partial \omega}{\partial z}\bigg|_{\beta=0} = 0$$

$$\frac{\partial \omega}{\partial z}\bigg|_{\beta=\pi} = \sigma(\alpha, \varphi) + \left(x \frac{\partial^2 \Phi_1}{\partial z^2} + y \frac{\partial^2 \Phi_2}{\partial z^2}\right)_{\beta=\pi} - 2\nu\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right)_{\beta=\pi} \tag{4.8}$$

Thus the derivation of function ω is reduced to the solution of the Neumann problem for the half-space.

From the additional condition (4.5), using (4.7), for the harmonic function Φ_4 we obtain

$$\Phi_4|_{\beta=\pi} = (1-2\nu)\,\omega|_{\beta=\pi} - (x\tau_{zx} + y\tau_{yz})_{\beta=\pi} \tag{4.9}$$

The second boundary condition for this function can be obtained by applying operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ to equation (4.4), rewritten as follows*:

$$\Phi_0|_{\beta=0} = \mathrm{Re} \sum_{m=1}^{\infty} F_m \left[1 - \frac{m}{4(1-\nu)}\right] r^m e^{im\varphi} + \mathrm{const} \tag{4.10}$$

Considering that $\Phi_4 = \partial \Phi_0 / \partial z$, we get

$$\partial \Phi_4 / \partial z|_{\beta=0} = 0 \tag{4.11}$$

and subsequently function Φ_4 can be obtained by means of the Mehler-Fock

* Note that function Φ_0 is generally defined within an additive constant.

integral transformation (Section 3).

Note, that if function Φ_0 is now defined by equation*

$$\Phi_0 = \int_{\infty}^{z} \Phi_4 dz \tag{4.12}$$

then for $\beta = 0$ its value will generally be different from the right-hand side of (4.10) by some harmonic function of variables κ and y. Let us show that by a judicious choice of coefficients F_m we can satisfy condition (4.10). Let us assume

$$\Phi_k = \Phi_k^{(0)} + \Phi_k' \qquad (k = 1, 2, 3, 4) \tag{4.13}$$

and let us define functions $\Phi_k^{(0)}$, containing coefficients F_m.

For $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ we have the following boundary conditions:

$$4(1-\nu)\Phi_1^{(0)}|_{\beta=0} = \operatorname{Re}\sum_{m=1}^{\infty} mF_m r^{m-1} e^{i(m-1)\varphi}, \qquad \left.\frac{\partial \Phi_1^{(0)}}{\partial z}\right|_{\beta=\pi} = 0$$

$$4(1-\nu)\Phi_2^{(0)}|_{\beta=0} = \operatorname{Re} i\sum_{m=1}^{\infty} mF_m r^{m-1} e^{i(m-1)\varphi}, \qquad \left.\frac{\partial \Phi_2^{(0)}}{\partial z}\right|_{\beta=\pi} = 0 \tag{4.14}$$

Applying the Mehler-Fock transformation, we can obtain the following expression for $\Phi_1^{(0)}$:

$$4(1-\nu)\Phi_1^{(0)} = \frac{2}{\pi}\sqrt{\lambda+\cos\beta}\,\operatorname{Re}\sum_{m=0}^{\infty}(-1)^m(m+1)F_{m+1}\times \tag{4.15}$$

$$\times\frac{2^{2m}a^m m!}{(2m)!}(\lambda^2-1)^{1/2\, m} e^{im\varphi}\times$$

$$\times\frac{\partial^m}{\partial\lambda^m}\left[\frac{1}{\sqrt{\lambda+\cos\beta}}\arctan\sqrt{\frac{\lambda+\cos\beta}{1-\cos\beta}}\right] \qquad (\lambda = \operatorname{ch}\alpha)$$

and a similar value for $\Phi_2^{(0)}$.

Further from (4.8) we find for $\omega^{(0)} = 2(1-\nu)\,\Phi_3^{(0)} - \Phi_4^{(0)}$

$$\left.\frac{\partial\omega^{(0)}}{\partial z}\right|_{\beta=0} = 0, \qquad -8\pi a^2(1-\nu)\left.\frac{\partial\omega^{(0)}}{\partial z}\right|_{\beta=\pi} =$$

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* If integral (4.12) diverges, then function Φ_0 must be obtained from the values of its first derivatives.

$$= \operatorname{sh}\tfrac{1}{2}\alpha\,(\operatorname{sh}^2\tfrac{1}{2}\alpha + 1 - 2\nu)\operatorname{Re}\sum_{m=1}^{\infty} m\gamma_m F_m \operatorname{th}^m \tfrac{1}{2}\alpha e^{im\varphi} \tag{4.16}$$

Here

$$\gamma_m = \frac{a^m 2^{2m}(m-1)!^2}{(2m-2)!} \tag{4.17}$$

Using the usual method based on the Mehler-Fock expansion, as well as particular solutions of type (4.3), we find

$$\omega^{(0)} = \omega^{*} + \bar{\omega} \tag{4.18}$$

where

$$-8\pi a\sqrt{2}(1-\nu)\,\omega^{*} = \sqrt{\operatorname{ch}\alpha + \cos\beta}\,\cos\tfrac{1}{2}\beta\operatorname{Re}\sum_{m=1}^{\infty} mE_m\gamma_m \operatorname{th}^m\tfrac{1}{2}\alpha e^{im\varphi} \tag{4.19}$$

$$8\pi a\sqrt{2\pi}(1-\nu)\,\bar{\omega} = (1-2\nu)\sqrt{\operatorname{ch}\alpha + \cos\beta}\,\frac{\cos^{1/2}\beta}{\operatorname{sh}^{2}{}^{1/2}\alpha}\times$$

$$\times \operatorname{Re}\sum_{m=1}^{\infty}\frac{m!\gamma_m}{\Gamma(m+{}^1/{}_2)}F_m \operatorname{th}^m\tfrac{1}{2}\alpha e^{im\varphi}\int_0^{\pi/2}\frac{\cos^{2m}\theta}{\cos^2{}^1/{}_2\beta/\operatorname{sh}^2{}^1/{}_2\alpha + \sin^2\theta}\,d\theta \tag{4.20}$$

and the integrals in (4.20) can be expressed in explicit form.

Now let us define function $\Phi_4^{(0)}$ from conditions

$$\Phi_4^{(0)}\big|_{\beta=\pi} = (1-2\nu)\,\omega^{(0)}\big|_{\beta=\pi}, \qquad \left.\frac{\partial\Phi_4^{(0)}}{\partial z}\right|_{\beta=0} = 0 \tag{4.21}$$

and noting that $\omega^{*}\big|_{\beta=\pi} = 0$, $\left.\dfrac{\partial\omega^{*}}{\partial z}\right|_{\beta=0} = 0$, we immediately get

$$\Phi_4^{(0)} = (1-2\nu)\bar{\omega} + A^{*}\omega^{*} \tag{4.22}$$

where A^* is as yet an arbitrary constant.

Investigation of the behavior of displacement w at $\alpha \to \infty$ shows that for the continuity of w at $\alpha \to \infty$ it is necessary* to assume $A^* = -1$. It remains to compute the value of the integral

$$\int_\infty^z \Phi_4\,dz\,\big|_{\beta=0} = \int_\infty^z [(1-2\nu)\bar{\omega} - \omega^{*}]\,dz\,\big|_{\beta=0} + \left.\int_\infty^z \Phi_4'\,dz\right|_{\beta=0} \tag{4.23}$$

and equate it to the right-hand side of equation (4.10).

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* Displacements u and v at $\alpha \to \infty$ are continuous for any value of A^*.

In succession we obtain

$$\int_{\infty}^{z} \bar{\omega}\, dz \Big|_{\beta=0} = -\frac{1-2\nu}{8(1-\nu)} \operatorname{Re} \sum_{m=1}^{\infty} F_m r^m e^{im\varphi} \tag{4.24}$$

$$\int_{\infty}^{z} \omega^* \, dz \Big|_{\beta=0} = -\frac{1}{4(1-\nu)} \operatorname{Re} \sum_{m=1}^{\infty} F_m \left(m - \tfrac{1}{2}\right) r^m e^{im\varphi} \tag{4.25}$$

$$\int_{\infty}^{z} \Phi_4' \, dz \Big|_{\beta=0} = \operatorname{Re} \sum_{m=0}^{\infty} G_m r^m e^{im\varphi} \tag{4.26}$$

where G_m are known numbers.

Substituting (4.24) through (4.26 into (4.10), we finally obtain

$$F_m = \frac{2}{2-\nu} G_m \qquad (m \geq 1) \tag{4.27}$$

which completes the general solution of the problem.

5. Example. Let two normal, concentrated, equally directed loads P (along the z-axis) be applied at points $\alpha = \alpha_0$, $\beta = \pm\pi$, $\phi = 0$ (Fig. 3).

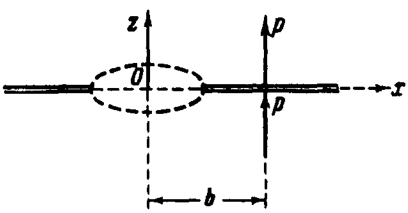

Fig. 3.

As $\tau_x = \tau_y = 0$, it follows

$$\Phi_1' = \Phi_2' = 0, \qquad \Phi_4' = (1-2\nu)\,\omega'$$

and for the function ω' the boundary conditions are:

$$\frac{\partial \omega'}{\partial z}\Big|_{\beta=0} = 0, \qquad \frac{\partial \omega'}{\partial z}\Big|_{\beta=\pi} = -\sigma(\alpha, \varphi) \tag{5.1}$$

Evidently the solution of the Neumann problem for this concentrated load is the function

$$\omega' = \frac{P}{2\pi\rho}, \qquad \rho = \sqrt{(x-b)^2 + y^2 + z^2} \qquad \left(b = a \operatorname{cth} \frac{1}{2}\alpha_0\right) \tag{5.2}$$

Thus, to find the coefficients G_m, it suffices to expand the function

$$\int (dz/\rho)\,|_{\beta=0}$$

into a trigonometric series in angle ϕ.

Using the familiar relation

$$\ln \sqrt{r^2 - 2br\cos\varphi + b^2} = -\sum_{m=1}^{\infty} \left(\frac{r}{b}\right)^m \frac{\cos m\varphi}{m} + \text{const} \qquad (5.3)$$

from (4.26) and (4.27) we immediately obtain

$$F_m = -\frac{(1-2\nu)P}{\pi(2-\nu)mb^m} \qquad (m \geqslant 1) \qquad (5.4)$$

Now functions $\Phi_1^{(0)}$, $\Phi_2^{(0)}$, $\Phi_3^{(0)}$ and $\Phi_4^{(0)}$ can be considered known [see equations (4.15), (4.18) to (4.22)].

Below is the formula for the stress distribution in the mid-section $z = 0$, $r < a$:

$$(\tau_{zx} - i\tau_{yz})_{\beta=0} = \frac{1-2\nu}{\pi^2(2-\nu)} \frac{P}{b\sqrt{a^2-r^2}} \frac{1}{1-\zeta}\left[1 + \sqrt{\frac{\zeta}{1-\zeta}} \operatorname{arc\,tg} \sqrt{\frac{\zeta}{1-\zeta}}\right] \qquad (5.5)$$

$$\zeta = \frac{x+iy}{b}$$

In conclusion let us note that the method suggested can be applied to the solution of the corresponding problems for an infinite elastic body weakened by an internal circular crack under most general external loading conditions.

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